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First results for the Coulomb gauge integrals using NDIM

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Abstract. The Coulomb gauge has at least two advantages over other gauge choices in that bound states between quarks and studies of confinement are easier to understand in this gauge. However, perturbative calculations, namely Feynman loop integrations, are not well defined (there are the so-called energy integrals) even within the context of dimensional regularization. Leibbrandt and Williams proposed a possible cure to such a problem by splitting the space-time dimension into $D = \omega + \rho$, i.e., introducing a specific parameter ρ to regulate the energy integrals. The aim of our work is to apply the negative dimensional integration method (NDIM) to the Coulomb gauge integrals using the recipe of split-dimension parameters and present complete results – finite and divergent parts – to the one- and two-loop level for arbitrary exponents of the propagators and dimension.

1 Introduction

The perturbative approach in quantum field theory (QFT) was responsible for several breakthrough ideas in physics and mathematics. One of these is dimensional regularization [1], i.e., analytic continuation of the space-time dimension D into an extended domain that allows for complex values. Feynman loop integrals gained a solid theoretical foundation and the renormalization process became simpler than it was (one had to use cut-offs and so on). Of course, this is only a partial picture of it all, say, the covariant side of the coin.

In algebraic non-covariant gauges [2], on the other hand, with dimensional regularization one was able to control divergences, e.g., in the light-cone gauge, but the results were not physically acceptable. In other words, double poles did appear in one-loop integral calculations, and Wilson loops did not have the correct behavior [3]. These problems were first overcome with the advent of what is known as the Mandelstam–Leibbrandt (ML) prescription [4]. More recently, we have shown that in the NDIM approach we do not need to invoke any kind of prescription to perform Feynman loop integrals in this gauge [5].

Among the non-covariant gauges we also have the Coulomb gauge (often referred to as the radiation gauge), where confinement [6] and bound states [7] are easier to deal with, the ghost propagator has no pole and unitarity is manifest. However, in such a gauge, no further insight has been achieved with the standard dimensional regularization technique, because it presents a gauge boson propagator of the form

$$G_{\mu\nu}^{ab}(q) = -\frac{\mathrm{i}\delta^{ab}}{q^2}\left[\eta_{\mu\nu} + \frac{n^2}{\boldsymbol{q}^2}q_\mu q_\nu - \frac{q\cdot n}{\boldsymbol{q}^2}\left(q_\mu n_\nu + n_\mu q_\nu\right)\right], \quad \text{with}\, n_\mu = (1,0,0,0), \tag{1}$$

which generates loop integrals like

$$\int \frac{\mathrm{d}^D q}{q^2(\boldsymbol{q}-\boldsymbol{p})^2}, \tag{2}$$

where bold face letters stand for three-momentum vectors.

The integral over the fourth component (in Euclidean space) or zeroth component (in Minkowski's), the so-called energy integral is not defined even within the context of dimensional regularization. Doust and Taylor [8] discussed Coulomb gauge loop integrals and presented a possible remedy for this problem in terms of an interpolating gauge (between Feynman and Coulomb, see also [9]). Leibbrandt [10], again, and Williams, presented another approach for these ill-defined integrals, a procedure called *split-dimensional regularization*. Both parts, namely energy and three-momentum sectors, separately need to be dimensionally regularized, that is, one parameter only, D, is not sufficient to render the integrals well defined. To overcome this problem, they introduced another regulating parameter, i.e., split the dimensionality of space-time into two distinct sectors, namely, $D = 4 - 2\epsilon = \omega + \rho$ and the divergences contained in energy integrals are expressed as poles in ρ besides the usual ones in terms of ω, that is to say, the integration measure is written down as $\mathrm{d}^D q = \mathrm{d}^\omega \boldsymbol{q}\mathrm{d}^\rho q_4$.

In a series of papers [11], Leibbrandt studied Coulomb gauge integrals to one- and two-loop level (with Heinrich) and presented results for divergent parts of several of them. Our aim in this work is two-fold: show that NDIM

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is the most versatile technique to carry out loop integrals, whether they come from covariant or non-covariant gauges; and to present complete results for the Coulomb gauge integrals to one- and two-loop level for arbitrary exponents of propagators and dimension.

The outline for our paper is as follows: in Sect. 2 we consider scalar and tensorial Coulomb gauge integrals at one-loop level, while Sect. 3 is devoted to two-loop integrals and in Sect. 4 we present our conclusions. In the appendix we discuss some technical issues.

2 One-loop Coulomb gauge integrals

To show how NDIM can handle Coulomb gauge integrals with ease we consider in this section one-loop integrals. Recall that negative dimensional integration is equivalent to positive dimensional integration over Grassmannian variables [12] — a property demonstrated by Dunne and Halliday — and for this very reason, propagators are raised to positive powers (they appear in the numerator of integrands) and the usual variables become Grassmannian ones. Another important point is that in the NDIM context it is as simple to work with arbitrary exponents of propagators as if we choose particular values for them. This is why we consider the general case. It is also worth remembering that for some types of diagrams, e.g., box integrals [13,14], there are divergences that are not related to the space-time dimension D, but appear as poles for particular values of exponents, say i and j, of the propagators, yielding singularities expressed as for instance $\Gamma(i-j)$, see also [15]. So, within the NDIM approach we can also trace back the origin of divergences.

The first two integrals we choose to work with are scalar ones,

$$g_1(i,j,k) = \int \mathrm{d}^D q (q^2)^i (q+p)^{2j} (\boldsymbol{q}^2)^k, \tag{3}$$

$$g_2(i,j,k) = \int \mathrm{d}^D q (q^2)^i (\boldsymbol{q}+\boldsymbol{p})^{2j} (\boldsymbol{q}^2)^k, \tag{4}$$

where generating functions for these, referred to as Coulomb gauge integrals, are

$$G_1 = \int \mathrm{d}^D q \exp\left[-\alpha q^2 - \beta(q+p)^2 - \gamma \boldsymbol{q}^2\right], \tag{5}$$

$$G_2 = \int \mathrm{d}^D q \exp\left[-\alpha q^2 - \beta(\boldsymbol{q}+\boldsymbol{p})^2 - \gamma \boldsymbol{q}^2\right], \tag{6}$$

with $D=\omega+\rho=4-\epsilon$ and $\mathrm{d}^D q = \mathrm{d}^\omega \boldsymbol{q}\,\mathrm{d}^\rho q_4$, in Euclidean space, following the split-dimension recipe of Leibbrandt et al.

Completing the square we can easily carry the integration out to get

$$G_1 = \left(\frac{\pi}{\alpha+\beta}\right)^{\rho/2} \left(\frac{\pi}{\lambda_1}\right)^{\omega/2} \exp\left[-\frac{(\alpha+\gamma)\beta\boldsymbol{p}^2}{\lambda_1}\right] \times \exp\left(-\frac{\alpha\beta p_4^2}{\alpha+\beta}\right), \tag{7}$$

$$G_2 = \left(\frac{\pi}{\alpha}\right)^{\rho/2} \left(\frac{\pi}{\lambda_1}\right)^{\omega/2} \exp\left[-\frac{(\alpha+\gamma)\beta\boldsymbol{p}^2}{\lambda_1}\right], \tag{8}$$

where $\lambda_1 = \alpha+\beta+\gamma$.

Taylor expanding both expressions in (5) and (7) we get the NDIM solutions for g_1 and g_2 by solving systems of linear algebraic equations.

The system of linear algebraic equations for the first integral is given by a 5×8 matrix,

$$\begin{cases} X_{13}+Y_{14} = i, \\ X_{123}+Y_{25} = j, \\ X_2+Y_3 = k, \\ Y_{123} = -X_{12}-\omega/2, \\ Y_{45} = -X_3-\rho/2, \end{cases}$$

with five equations and eight "unknowns", corresponding to the various summation indices coming from the Taylor and multinomial expansions. They are solvable only within the lower quadratic 5×5 dimension matrices. There are a grand total of 56 possible square matrices of this type (i.e., 5×5) from which 36 yield relevant non-vanishing and workable solutions while the remaining 20 yield a set of trivial solutions (i.e., the related systems do not have a solution). We know from our previous works (see for instance [13]) that all those non-trivial solutions will generate power series of the hypergeometric type, known as hypergeometric functions [16]. Moreover, all of them are related by analytic continuation, either directly or indirectly. In our present case, namely the integral g_1, there are triple as well as double series, among which we choose to consider only the simplest ones,

$$\begin{aligned} &g_1^{A[\mathrm{AC}]}(i,j,k) \\ &= f_1^{A,[\mathrm{AC}]} \sum_{n_1,n_2,n_3=0}^{\infty} \left(\frac{\boldsymbol{p}^2}{p_4^2}\right)^{n_{123}} \Big[\Big((-1)^{n_2}(-i|n_{123}) \\ &\quad \times (k+\omega/2|n_{12})(D/2+j|n_{23})(1-i-\rho/2|n_{123})\Big) \\ &\quad \Big/ \Big(n_1!n_2!n_3!(1+j+k+D/2|n_{23}) \\ &\quad \times (1-i-\rho/2|n_{12})(\sigma+\omega/2-i|n_{123})\Big)\Big], \end{aligned} \tag{9}$$

where the superscript "[AC]" means analytic continuation (to the positive dimensional region) and we define the shorthand notation $n_{AB} = n_A+n_B$, while $(x|y) \equiv (x)_y = \Gamma(x+y)/\Gamma(x)$ is the Pochhammer symbol and

$$\begin{aligned} f_1^{A,[\mathrm{AC}]} &= \pi^{D/2}(-p_4^2)^i (\boldsymbol{p}^2)^{j+k+D/2} (-j|j+k+\omega/2) \\ &\quad \times (-k|j+k+D/2)(-i+\sigma+\omega/2|i-\sigma-\omega/ \\ &\quad 2-D/2-j-k), \end{aligned} \tag{10}$$

where $\sigma = i+j+k+D/2$. Observe that the above result is valid for negative j,k. Among the 36 possible series this is the only one that has the form $\Sigma(\cdots)^{a+b+c}$, where the $\cdots$ stands for the specific kinematical configuration.

For the case of double series we have, e.g.,

$$\begin{aligned} &g_1^{B,[\mathrm{AC}]}(i,j,k) \\ &= f_1^{B,[\mathrm{AC}]} \sum_{n_1,n_2=0}^{\infty} \left(\frac{\boldsymbol{p}^2}{p_4^2}\right)^{n_{12}} \Big[\Big((-\sigma|n_{12})(-k|n_2) \end{aligned}$$

$$\times(\omega/2+k|n_1)(1-k-\sigma-D/2|n_2)\Big) \Big/\Big(n_1!n_2!(\omega/2|n_{12})(1-j-k-D/2|n_2)\Big)\Big], \quad (11)$$

where

$$f_1^{B,[\mathrm{AC}]} = \pi^{D/2}(p_4^2)^\sigma(-i|\sigma)(-j|\sigma)(\omega/2|k) \times (k+\sigma+D/2|-2\sigma-k-D/2), \quad (12)$$

and the result is valid when i, j are negative.

This hypergeometric series representation is four-fold degenerate. Here we mention an important point in the process of analytic continuation to positive dimension and negative values of exponents of propagators referred to above. The result for the negative dimensional space region for g_1^B is in fact given by a sum of two terms – the second one being also four-fold degenerate; however, when we perform the analytic continuation, this second term vanishes, because it contains a factor of the form (forgetting about the minus sign)

$$\frac{1}{(1|-\rho/2)} = \frac{\Gamma(1)}{\Gamma(1-\rho/2)} \xrightarrow{\mathrm{AC}} (0|\rho/2) = \frac{\Gamma(\rho/2)}{\Gamma(0)} = 0,$$

which always vanishes (see also [18]) since $\rho \neq 0$ by definition and where, as usual in the NDIM approach we make use of the Pochhammer symbol property $(a|b) = (-1)^b/(1-a|-b)$.

To close this part of the computation, we just mention that of course there are other hypergeometric series which represent the same Feynman integral in other kinematical regions, e.g., there is a double series of the form,

$$\sum_{a,b=0}^{\infty} \frac{\Gamma(...)}{a!b!}\left(\frac{p_4^2}{\boldsymbol{p}^2}\right)^a,$$

that is, one of the series (with summation index b) has unit argument and can be recast as a ${}_3F_2(...|1)$.

These are just a few different manners in which we may write down the result for the integral g_1.

The second integral is easier than the first, and its result is also degenerate, i.e., there are a total of five 4×4 systems to be solved of which one has no solution and the remaining four, after properly summed give the same result, yielding

$$g_2 = (-\pi)^{D/2}(\boldsymbol{p}^2)^\sigma\Big[\Big(\Gamma(1+i)\Gamma(1+j)\Gamma(1-\sigma-\omega/2) \times\Gamma(1+i+k+\rho/2)\Big)\Big/\Big(\Gamma(1+\sigma)\Gamma(1+i+\rho/2) \times\Gamma(1-i-k-D/2) \quad \times\Gamma(1-j-\omega/2)\Big)\Big], \quad (13)$$

which after analytically continuing to positive D becomes

$$g_2^{[\mathrm{AC}]}(i,j,k) = \pi^{D/2}(\boldsymbol{p}^2)^\sigma(\sigma+\omega/2|-2\sigma-\omega/2) \times(-i|-\rho/2)(-j|\sigma)(-i-k-\rho/2|\sigma). \quad (14)$$

Next we consider Feynman integrals in the Coulomb gauge with tensorial structures. Again, in the same way that we treated in [17] the case of the covariant gauge, we show here how NDIM can handle these Coulomb gauge tensorial integrals in a similar manner. Let

$$g_3(i,j,k) = \int \mathrm{d}^D q(q^2)^i(\boldsymbol{q}+\boldsymbol{p})^{2j}(2\boldsymbol{q}\cdot\boldsymbol{p})^k, \quad (15)$$

and

$$g_4(i,j,k,m) = \int \mathrm{d}^D q(q^2)^i(\boldsymbol{q}+\boldsymbol{p})^{2j}(\boldsymbol{q}^2)^k(2\boldsymbol{q}\cdot\boldsymbol{p})^m, \quad (16)$$

so that, after some algebraic manipulations, we eventually get the result

$$g_3^{[\mathrm{AC}]}(i,j,k) = \pi^{D/2}(-2)^k(\boldsymbol{p}^2)^\sigma(-i|-j-D/2) \times(\sigma+\omega/2|j-\sigma)(-j|\sigma){}_3F_2(\{3\}|1), \quad (17)$$

where the set of parameters $\{3\} \equiv \{a_3, b_3, c_3; e_3, f_3\}$ for the hypergeometric function ${}_3F_2(\{3\})$ is given in Table 1.

Table 1. Parameters for hypergeometric functions in (17) and (18)

Parameters	${}_3F_2(\{3\}\|1)$	${}_3F_2(\{4\}\|1)$
a	$-k/2$	$-m/2$
b	$1/2-k/2$	$1/2-m/2$
c	$j+\omega/2$	$j+\omega/2$
e	$1+i+j+D/2$	$1+i+j+k+D/2$
f	$1-i-k-D/2$	$1-i-k-m-D/2$

We must observe here that for (15) the exponent $k \geq 0$ always, and in (16) the exponent $m \geq 0$ always, and these must not be analytically continued into the region of negative values, whereas the exponents i, j do follow the usual analytic continuation process to get the final result for the integrals.

From our previous work [17] on the NDIM approach to tensorial integrals, we know that the best solution for such a kind of integrals is a truncated hypergeometric function, because it contains all the cases of interest in the same formula: scalar, vector and arbitrary tensor rank. The hypergeometric function above is clearly truncated for positive integers k. This result, among the five possible hypergeometric series representations of such an integral, is the only one that is a truncated series for even and odd values of the propagator exponent k, since it assumes only positive values.

Finally, the result for the tensorial integral with three propagators,

$$g_4^{[\mathrm{AC}]}(i,j,k,m) = \pi^{D/2}(-2)^m(\boldsymbol{p}^2)^{\sigma'}(-i|-\rho/2) \times(\sigma'+\omega/2|j-\sigma')(-j|\sigma') \times(-i-k-\rho/2|-j-\omega/2){}_3F_2(\{4\}|1), \quad (18)$$

where $\sigma' = \sigma+m = i+j+k+m+D/2$. Note that the result (18) contains the previous one, (17), in the particular case when $k=0$; it is valid also when i, j, k are negative and m positive. The five parameters $\{4\} \equiv \{a_4, b_4, c_4; e_4, f_4\}$ are given in the table, and clearly the hypergeometric function is also truncated for even and odd positive integers m.

The well-known hypergeometric function ${}_3F_2$ is defined by the series

$${}_3F_2\left[\begin{array}{c} a,b,c \\ e,f \end{array}\middle| z\right] = \sum_{n=0}^{\infty} \frac{(a|n)(b|n)(c|n)}{(e|n)(f|n)} \frac{z^n}{n!},$$

so, when we refer to ${}_3F_2(...|1)$ we are meaning the above series. For more details on hypergeometric functions the reader is referred to, e.g., [16].

3 Two-loop Coulomb gauge integrals

As far as we know, NDIM is the only approach where Feynman integrals in different gauges, covariant and non-covariant alike, can be neatly performed, without reference to any special prescription to handle peculiar non-covariant singularities in the boson propagator. In the usual covariant gauges several calculations were carried out, e.g., the one-loop n-point function [18], scalar integrals for photon–photon scattering in QED [13] and genuine [19] two-loop three-point integrals. On the non-covariant side, we have gotten an important original result: Light-cone integrals in the NDIM context do not need the famous ML-prescription [4] to circumvent the gauge dependent singularities [5], as well as avoiding other features which turn the calculation cumbersome – such as using partial fractioning [20] (a mandatory feature there) and integration over components.

Coulomb gauge two-loop integrals can be treated within the NDIM methodology as well. Consider, for example,

$$I_1(i,j,k,m) = \int \mathrm{d}^D q \mathrm{d}^D r (q^2)^i (r^2)^j (p-r-q)^{2k} (\boldsymbol{r}^2)^m, \tag{19}$$

$$I_2(i,j,k,m) = \int \mathrm{d}^D q \mathrm{d}^D r (q^2)^i (q-r)^{2j} (\boldsymbol{r}^2)^k (\boldsymbol{p}-\boldsymbol{q})^{2m}, \tag{20}$$

which can be generated by

$$\mathcal{I}_1 = \int \mathrm{d}^D q \mathrm{d}^D r \exp\left[-\alpha q^2 - \beta r^2 - \gamma (p-r-q)^2 - \theta \boldsymbol{r}^2\right], \tag{21}$$

$$\mathcal{I}_2 = \int \mathrm{d}^D q \mathrm{d}^D r \exp\left[-\alpha q^2 - \beta (q-r)^2 - \gamma \boldsymbol{r}^2 - \theta (\boldsymbol{p}-\boldsymbol{q})^2\right]. \tag{22}$$

Following the usual steps of NDIM [21], we get for the first integral a 6×11 matrix for the system of linear algebraic equations to be solved:

$$\left\{ \begin{array}{c} X_{123} + Y_{12467} = i, \\ X_{13} + Y_{2368} = j, \\ X_{123} + Y_{13578} = k, \\ X_2 + Y_{45} = m, \\ X_{12} + Y_{12345} = -\omega/2, \\ X_3 + Y_{678} = -\rho/2, \end{array} \right. \tag{23}$$

which generates 462 (6×6) possible hypergeometric series representations for the integral in question. Of these, 216 have solutions in terms of hypergeometric series whose variable is either $z = p_4^2/\boldsymbol{p}^2$ or z^{-1}. Among these power series the simplest ones are double series which we consider in more detail. Of course, there are also series of the following form:

$$\sum_{a,b,c,e=0}^{\infty} \frac{z^{a+b}}{a!b!} \frac{(z^{-1})^{c+e}}{c!e!} \frac{\Gamma(...)}{\Gamma(...)},$$
$$\sum_{a,b,c,e=0}^{\infty} \frac{z^{a}}{a!} \frac{(z^{-1})^{b+c+e}}{b!c!e!} \frac{\Gamma(...)}{\Gamma(...)}, \tag{24}$$

which can only be convergent if $z = z^{-1} = 1$. Since this is a particular case, where $\boldsymbol{p}^2 = p_4^2$, we will not study it.

Let us consider the solution written in terms of double hypergeometric series,

$$\begin{aligned} & I_1^{A,[\mathrm{AC}]}(i,j,k,m) \\ & = \pi^D (p_4^2)^{\sigma''} P_A^{[\mathrm{AC}]} \sum_{n_1,n_2=0}^{\infty} \Big[\Big((-\sigma''|n_{12})(-m|n_2) \\ & \quad \times (m+\omega/2|n_1)(1-m-\sigma''-D/2|n_2) \Big) \\ & \quad \Big/ \Big(n_1! n_2! (1-i-k-m-D/2|n_2)(\omega/2|n_{12}) \Big) \Big] \\ & \quad \times \left(\frac{\boldsymbol{p}^2}{p_4^2}\right)^{n_{12}}, \end{aligned} \tag{25}$$

where $\sigma'' = \sigma' + D/2 = i+j+k+m+D$ and $P_A^{[\mathrm{AC}]}$ is a product of Pochhammer symbols,

$$\begin{aligned} P_A^{[\mathrm{AC}]} & = (-i|i+k+D/2)(-k|i+k+D/2)(\omega/2|m) \\ & \quad \times (i+k+D|m)(-j|j-\sigma'') \\ & \quad \times (\sigma''+m+D/2|j-\sigma''), \end{aligned} \tag{26}$$

where the exponents of the propagators i,j,k must assume negative values. There is another double series, in the other kinematical region, where $|p_4^2/\boldsymbol{p}^2| < 1$, namely,

$$\begin{aligned} & I_1^{B[\mathrm{AC}]}(i,j,k,m) \\ & = \pi^D (\boldsymbol{p}^2)^{\sigma''} P_B^{[\mathrm{AC}]} \sum_{n_i=0}^{\infty} \Big[\Big((-\sigma''|n_2) \\ & \quad \times (-j-m-\omega/2|n_2-n_1)(m+\omega/2|n_1) \\ & \quad \times (j+m+D/2|n_1) \Big) \Big/ \Big(n_1! n_2! (1-\sigma''-\omega/2|n_2-n_1) \\ & \quad \times (\omega/2|n_1) \Big) \Big] \left(\frac{p_4^2}{\boldsymbol{p}^2}\right)^{n_{12}}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} P_B^{[\mathrm{AC}]} & = (-i|i-\sigma'')(-j|-m-\omega/2) \\ & \quad \times (-k|j+k+m+D/2) \\ & \quad \times (\omega/2|m)(i+k+D|-k-D/2) \\ & \quad \times (\rho/2|k+\omega/2). \end{aligned} \tag{28}$$

The second integral is much simpler than the former, in that the hypergeometric series representations involved are all summable. Summing them is an easy task and the result can be written in terms of gamma functions:

$$\begin{aligned}I_2^{[\mathrm{AC}]}(i,j,k,m) &= \pi^D(\boldsymbol{p}^2)^{\sigma''}(-i|-\rho/2)\\ &\times(\sigma''+\omega/2|-2\sigma''-\omega/2)(-k|2k+\omega/2)\\ &\times(-m|2m+\omega/2)(-j|i+2j+k+D)\\ &\times(-i-j-k-D/2-\rho/2|i+\rho/2)\\ &\times(j+k+D-\rho/2|-k-\omega/2). \end{aligned}\tag{29}$$

This is a 36-fold degenerate result, i.e., of the overall 56 (5×5) systems, 36 ones have non-trivial solutions, which after being summed (see e.g. [21]) and analytically continued give (29). It is important to note that this result allows for negative values for (i,j,k,m) and positive ones for D. For negative m, see the appendix.

4 Conclusion

Splitting the space-time dimension D in the dimensional regularization context into an energy sector and a momentum sector, each with a specific regularizing parameter, it was possible for Leibbrandt et al. to perform perturbative calculations in the Coulomb gauge at one- and two-loop level. However, the calculations are very involved and they were able to present explicit results only for the divergent parts of the integrals. On the other hand, using NDIM we showed here that we can calculate complete results for the same integrals, and not only that; they did not have to be carried out separately. In our approach we can consider several of them at the same time, because we leave the exponents of the propagators arbitrary, the integrals being either scalar or tensorial.

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Appendix A: Special cases: extracting poles

To make things a little more illuminating we consider in this appendix some technical issues relevant to the results for sample special cases.

A.1 One loop

Let us consider for instance the particular case where the exponents of propagators in the integral (16) are $i=j=-1, k=-2, m=2$. The result for this integral is obtained from (18),

$$\begin{aligned}g_4^{[\mathrm{AC}]}(-1,-1,-2,2) &= 4\pi^{D/2}(\boldsymbol{p}^2)^{D/2-2}\\ &\times\frac{\Gamma(1-\rho/2)\Gamma(\omega/2-1)\Gamma(3-D/2)\Gamma(D/2-1)}{\Gamma(3-\rho/2)\Gamma(D/2-2+\omega/2)}\\ &\times{}_3F_2\left(\left.\begin{matrix}-1,-1/2,\omega/2-1\\ D/2-3,2-D/2\end{matrix}\right|1\right),\end{aligned}\tag{A1}$$

observing that now $\sigma'=D/2-2$. We proceed as usual in dimensional regularization, taking $D=4-2\epsilon$, $\omega=3-\epsilon$, $\rho=1-\epsilon$ and Taylor expanding around $\epsilon=0$, to get

$$\begin{aligned}g_4^{[\mathrm{AC}]}(-1,-1,-2,2) &= 4\pi^{2-\epsilon}(\boldsymbol{p}^2)^{-\epsilon}\\ &\times\left[\frac{8}{3}+\left(-\frac{8\gamma_{\mathrm{E}}}{3}+\frac{64}{9}-\frac{16}{3}\ln 2\right)\epsilon+\mathcal{O}(\epsilon^2)\right]\\ &\times{}_3F_2\left(\left.\begin{matrix}-1,-1/2,\omega/2-1\\ D/2-3,2-D/2\end{matrix}\right|1\right),\end{aligned}\tag{A2}$$

where γ_{E} is the Euler constant.

Now we turn to the theory of hypergeometric functions [16]. When a numerator parameter is a negative integer the series is a truncated one. This is exactly our case, and the hypergeometric function above has only two terms,

$$\begin{aligned}&{}_3F_2\left(\left.\begin{matrix}-1,-1/2,\omega/2-1\\ D/2-3,2-D/2\end{matrix}\right|1\right)\\ &=1+\frac{(-1|1)(-1/2|1)(\omega/2-1|1)}{1!(D/2-3|1)(2-D/2|1)}\\ &=1-\frac{(1-\epsilon)}{4\epsilon(1+\epsilon)},\end{aligned}\tag{A3}$$

which substituted into (A2) yields

$$\begin{aligned}&g_4^{[\mathrm{AC}]}(-1,-1,-2,2)\\ &=4\pi^{2-\epsilon}(\boldsymbol{p}^2)^{-\epsilon}\times\left(-\frac{2}{3\epsilon}+\frac{20}{9}+\frac{2\gamma_{\mathrm{E}}}{3}+\frac{4\ln 2}{3}\right).\end{aligned}\tag{A4}$$

We remember that the original integral was

$$\begin{aligned}g_4(-1,-1,-2,2) &= \int\frac{\mathrm{d}^Dq(2\boldsymbol{q}\cdot\boldsymbol{p})^2}{q^2(\boldsymbol{q}+\boldsymbol{p})^2(\boldsymbol{q}^2)^2}\\ &=4p_gp_h\int\frac{\mathrm{d}^Dq q^g q^h}{q^2(\boldsymbol{q}+\boldsymbol{p})^2(\boldsymbol{q}^2)^2}\\ &=4p_gp_h g_4^{gh},\end{aligned}\tag{A5}$$

where

$$g_4^{gh}=\int\frac{\mathrm{d}^Dq q^g q^h}{q^2(\boldsymbol{q}+\boldsymbol{p})^2(\boldsymbol{q}^2)^2},\tag{A6}$$

so that we obtain the final result:

$$\begin{aligned}g_4^{[\mathrm{AC}],gh}(-1,-1,-2,2) &= \pi^{2-\epsilon}(\boldsymbol{p}^2)^{-1-\epsilon}\delta^{gh}\\ &\times\left(-\frac{2}{3\epsilon}+\frac{20}{9}+\frac{2\gamma_{\mathrm{E}}}{3}+\frac{4\ln 2}{3}\right).\end{aligned}\tag{A7}$$

A.2 Two loops

Let us consider two particular cases for integral (20): The first one where $i=k=-2, j=-1, m=1$ and a second one where $i=k=-2, j=m=-1$.

Observe that in the first case there is one exponent, m, which is positive, so we must not analytically continue it (see for instance our previous papers [5,17]). The related Pochhammer symbol $(-m|2m+\omega/2)$ was generated by

$$\frac{\Gamma(1+m)}{\Gamma(1-m-\omega/2)} = \frac{1}{\Gamma(1+m|-2m-\omega/2)} \stackrel{\text{AC}}{=} (-1)^{2m+\omega/2}(-m|2m+\omega/2).$$

However, it must not be analytically continued since we are interested in the special case where m is positive ($m = 1$). So the result for (20), which allows one to take m positive, now reads

$$\begin{aligned} & I_2^{[\text{AC}]}(i,j,k,m) \\ & = \pi^D(\boldsymbol{p}^2)^{\sigma''}(-i|-\rho/2)(\sigma''+\omega/2|-2\sigma''-\omega/2) \\ & \quad \times(-k|2k+\omega/2)(-j|i+2j+k+D) \\ & \quad \times(-i-j-k-D/2-\rho/2|i+\rho/2) \\ & \quad \times(j+k+D-\rho/2|-k-\omega/2) \\ & \quad \times\frac{\Gamma(1+m)}{\Gamma(1-m-\omega/2)}(-1)^{2m+\omega/2}. \end{aligned} \tag{A8}$$

Now it is easy to expand (we use the MAPLE V software) around $D = 4 - 2\epsilon$, with $\sigma'' = D - 4$, to obtain the poles, double and simple ones, plus finite part

$$\begin{aligned} & I_2^{[\text{AC}]}(-2,-1,-2,1) \\ & = -i^{1-\epsilon}\pi^{4-2\epsilon}(\boldsymbol{p}^2)^{-2\epsilon}\left[-\frac{1}{2\epsilon^2}+\frac{7+12\ln 2+6\gamma_{\text{E}}}{6\pi\epsilon}\right. \\ & \quad +\frac{-520-112\gamma_{\text{E}}-192\gamma_{\text{E}}\ln 2+43\pi^2-224\ln 2}{48\pi} \\ & \quad \left.-\frac{48\gamma_{\text{E}}+192\ln^2 2}{48\pi}+\mathcal{O}(\epsilon)\right]. \end{aligned} \tag{A9}$$

The second special case can be studied using (29) directly since in this case all exponents ($i = k = -2, j = m = -1$) are negative, and ($\sigma'' = D - 6$). Using again MAPLE V software to expand around $\epsilon = 0$, we get a simple pole plus finite part,

$$\begin{aligned} & I_2^{[\text{AC}]}(-2,-1,-2,-1) \\ & = \pi^{4-2\epsilon}(\boldsymbol{p}^2)^{-2-2\epsilon}\left(\frac{1}{3\epsilon}+\frac{1}{3}-\frac{2\gamma_{\text{E}}}{3}-\frac{4\ln 2}{3}\right). \end{aligned} \tag{A10}$$

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